

LECTURE 28 BEGINS:

1 Vector Space

Let U and S be two non-empty sets. A mapping $f : U \times S \mapsto S$ is said to be an external composition of U with S . Each ordered pair $(u, s) \in U \times S$ has a definite image $f(u, s) \in S$. For example, let S be the set of all real matrices of 3×3 type, and U be the set of all real numbers $U = \mathbb{R}$. The mapping $*$: $U \times S \mapsto S$ defined by $C * A = CA \in S$, $C \in \mathbb{R}, A \in S$.

Let $V \neq \emptyset$ and $\oplus$ be a composition on V . Let $(\mathbb{F}, +, \cdot)$ be the field of scalars, and let $\odot$ be an external composition of $\mathbb{F}$ with V . Then, V is said to be vector space over the field $\mathbb{F}$, if the following conditions are true:

- (i) $\alpha \oplus \beta \in V, \forall \alpha, \beta \in V$ (closedness property).
- (ii) $\alpha \oplus \beta = \beta \oplus \alpha$ for $\alpha, \beta \in V$.
- (iii) $(\alpha \oplus \beta) \oplus \gamma = \alpha \oplus (\beta \oplus \gamma), \forall \alpha, \beta, \gamma \in V$.
- (iv) $\exists$ an element $\theta \in V$ such that $\alpha \oplus \theta = \alpha, \forall \alpha \in V$.
- (v) for each $\alpha \in V, \exists (-\alpha) \in V$ such that $\alpha \oplus (-\alpha) = \theta$.
- (vi) $c \odot \alpha \in V, \forall c \in \mathbb{F}, \alpha \in V$.
- (vii) $c \odot (d \odot \alpha) = (c \cdot d) \odot \alpha, \forall c, d \in \mathbb{F}, \alpha \in V$.
- (viii) $c \odot (\alpha \oplus \beta) = (c \odot \alpha) \oplus (c \odot \beta), \forall c \in \mathbb{F}, \alpha, \beta \in V$.
- (ix) $(c + d) \odot \alpha = (c \odot \alpha) \oplus (d \odot \alpha), \forall c, d \in \mathbb{F}, \alpha \in V$.
- (x) $1 \odot \alpha = \alpha$, where 1 denotes the multiplicative identity element.

Example 1. Let $V = \{(a_1, a_2, \dots, a_n) : a_i \in \mathbb{R}, \forall i = 1, 2, \dots, n\} = \mathbb{R}^n$ and $\mathbb{F} = \mathbb{R}$. The usual $\oplus$ is the vector addition, and the usual $+, \cdot$ are the the operations on field. Verify whether $(V, \oplus, \odot)$ is a vector space with respect to the field of scalars $\mathbb{R}$.

Solution. We start verifying the properties of the vector space stated earlier one-by-one. We choose α, β from V and find out that $\alpha + \beta \in V$ and $\alpha \oplus \beta = \beta \oplus \alpha$. Also, for $\gamma \in V, (\alpha \oplus \beta) \oplus \gamma = \alpha \oplus (\beta \oplus \gamma)$. Here we have the zero element $\theta = (0, 0, \dots, 0) \in \mathbb{R}^n$ which assures that $\alpha \oplus \theta = \alpha$. Similarly, there does exist $-\alpha \in V$ for all α such that $\alpha \oplus (-\alpha) = \theta$. In a similar fashion we deal with the rest of the properties, which is not so hard to check in this case.

Example 2. Let $V = \{A = (a_{ij}) : a_{ij} \in \mathbb{R}, 1 \leq i, j \leq 3\}$ be a set of all 3×3 matrices and $\mathbb{F} = \mathbb{R}$. With the usual matrix addition and scalar multiplication, verify that $(V, +, \cdot)$ is a vector space with respect to the field of scalars $\mathbb{R}$.

Solution. Do it by yourself.

2 Subspace

Let V be a vector space over a field of scalars $\mathbb{F}$ with respect to $\oplus$ and $\odot$. Let $W \subset V$. Then, W is said to be a subspace of V if W forms a vector space with respect to the same operations, i.e., $\oplus$ and $\odot$.

Example 1. Let $S = \{(x, y, z) \in \mathbb{R}^3 : y = z = 0\} \in \mathbb{R}^3$ and $\mathbb{F} = \mathbb{R}$. Prove that S is a subspace of $\mathbb{R}^3$.

Example 2. Let $S = \{(x, y, z) \in \mathbb{R}^3 : y > 0, z > 0\} \in \mathbb{R}^3$ and $\mathbb{F} = \mathbb{R}$. Prove that S is NOT a subspace of $\mathbb{R}^3$. (Hint: $\nexists \theta \in S$)

Example 3. Let $S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = z^2\} \in \mathbb{R}^3$ and $\mathbb{F} = \mathbb{R}$. Prove that S is NOT a subspace of $\mathbb{R}^3$. {Hint: Take $\alpha = (3, 4, 5)$, $\beta = (-6, 8, 10) \in S$ }

LECTURE 29 BEGINS:

3 Linear Combination

Let V be a vector space. Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be the elements in V . A vector $\beta \in V$ is said to be a linear combination of $\alpha_1, \alpha_2, \dots, \alpha_n$, if β can be expressed as

$$\beta = c_1\alpha_1 + c_2\alpha_2 + \dots + c_n\alpha_n = \sum_{i=1}^n c_i\alpha_i, \quad (1)$$

where $c_1, c_2, \dots, c_n$ are constants in $\mathbb{F}$.

Example 1. $\beta = (1, 1)$, $\alpha_1 = (\frac{1}{2}, \frac{1}{2})$ and $\beta_1 = (\frac{1}{2}, \frac{1}{2}) \implies \beta = \alpha_1 + \alpha_2$ OR $\alpha_1 = (\frac{1}{3}, \frac{3}{4})$ and $\beta_1 = (\frac{3}{4}, \frac{1}{4}) \implies \beta = \alpha_1 + \alpha_2$.

4 Linear Dependence and Independence of Vectors

Let $\alpha_1, \alpha_2, \dots, \alpha_n \in V$ and $c_1, c_2, \dots, c_n \in \mathbb{F}$ such that $c_1\alpha_1 + c_2\alpha_2 + \dots + c_n\alpha_n = \theta \in V$ with not all c_i 's zero. Such a set $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ is said to be linearly dependent.

If $c_1\alpha_1 + c_2\alpha_2 + \dots + c_n\alpha_n = \theta \in V$, only when $c_1 = c_2 = \dots = c_n = 0$, then the set $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ is said to be linearly independent.

Example 1. Check whether the set $\{(1, 1), (\frac{1}{2}, \frac{1}{2})\}$ is linearly dependent or not.

Solution. Let $\alpha_1 = (1, 1)$ and $\alpha_2 = (\frac{1}{2}, \frac{1}{2})$ such that

$$\begin{aligned} c_1\alpha_1 + c_2\alpha_2 &= \theta \\ \implies (c_1, c_1) + \left(\frac{c_2}{2}, \frac{c_2}{2}\right) &= (0, 0) \\ \implies c_1 + \frac{c_2}{2} &= 0, \quad c_1 + \frac{c_2}{2} = 0 \\ \implies c_1 + \frac{c_2}{2} &= 0 \implies c_1 = -\frac{c_2}{2}. \end{aligned}$$

Taking $c_1 = 1$, $c_2 = -2$, i.e., $\alpha_1 = -c_2\alpha_2 = 2\alpha_2$, which implies that $\{(1, 1), (\frac{1}{2}, \frac{1}{2})\}$ is linearly dependent.

Example 2. Check whether the set $\{(1, 0), (0, 1)\}$ is linearly dependent or not.

Solution. Let $\alpha_1 = (1, 0)$ and $\alpha_2 = (0, 1)$ such that

$$\begin{aligned} c_1\alpha_1 + c_2\alpha_2 &= \theta \\ \implies (c_1, 0) + (0, c_2) &= (0, 0) \\ \implies (c_1, c_2) &= (0, 0) \\ \implies c_1 = 0, \quad c_2 &= 0. \end{aligned}$$

This implies that $\{(1, 0), (0, 1)\}$ is linearly independent.

Example 3. Check whether the set $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ is linearly dependent or not.

Example 4. Check whether the set $\{(1, 2, 4), (2, 4, 8)\}$ is linearly dependent or not.

5 Linear Span of a Set

Let $V \neq \emptyset$ be a vector space and let $S = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be a set of vectors/elements in V . If the set S is such that every element of V is expressed as a linear combination of elements in S , then S is called the linear span of V .

6 Basis

Let $V \neq \emptyset$ be a vector space. A set $S \subset V$ is said to be a basis of V if

- (i) S is a linearly independent set.
- (ii) S generates V .

Example 1. Let $V = \mathbb{R}^3$ be the vector space with respect to the usual operations over $\mathbb{R}$. Let $S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\} \subset \mathbb{R}^3$. Verify whether S is a basis or not.

Solution. Let $c_1, c_2, c_3 \in \mathbb{F}$ such that $c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 = \theta = (0, 0, 0)$, which implies $c_1 = c_2 = c_3 = 0$. Hence, S is a linearly independent set.

Now let $\xi = (a, b, c) \in \mathbb{R}^3$, then we can write

$$\xi = (a, b, c) = a(1, 0, 0) + b(0, 1, 0) + c(0, 0, 1) = a\alpha_1 + b\alpha_2 + c\alpha_3.$$

This shows that S is a basis of V .

Example 2. Let $V = \mathbb{R}^3$ be the vector space with respect to the usual operations over $\mathbb{R}$. Let $S = \{(1, 2, 1), (2, 1, 1), (1, 1, 2)\} \subset \mathbb{R}^3$. Verify whether S is a basis of $\mathbb{R}^3$ or not.

Solution. Let $c_1, c_2, c_3 \in \mathbb{F}$ such that $c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 = \theta = (0, 0, 0)$, which implies $c_1 = c_2 =$

$$c_3 = 0$$

$$\begin{aligned}c_1 + 2c_2 + c_3 &= 0, \\2c_1 + c_2 + c_3 &= 0, \\c_1 + c_2 + 2c_3 &= 0, \\ \implies c_1 = c_2 = c_3 &= 0.\end{aligned}$$

Hence, S is a linearly independent set.

7 Dimension

The number of elements in a basis of a vector space is called the dimension of V (or rank of V), and it is denoted by $\dim V$ or $\dim(V)$.

Example 1. Note that the dimension of the vector space in the previous example is: $\dim(V) = 3$. Similarly, $\dim(\mathbb{R}^n) = n$.

Example 2. Let $W \subset \mathbb{R}^3$ given by $W = \{(x, y, z) \in \mathbb{R}^3 : x + 2y + z = 0, 2x + y + 3z = 0\}$. then, find the basis and dimension of W .

Solution. Let $\xi = (a, b, c) \in W^3$ such that

$$\begin{aligned}a + 2b + c &= 0, \\2a + b + 3c &= 0, \\ \implies \frac{a}{5} = \frac{b}{-1} = \frac{c}{-3} &= k(\text{say}) \\ \implies a = 5k, b = -k, c = -3k \\ \implies \xi = k(5, -1, -3), k \in \mathbb{R}.\end{aligned}$$

This means that $\xi = (a, b, c) = k(5, -1, -3), k \in \mathbb{R}$ belongs to W . In other words, $W = L(\alpha), \alpha(5, -1, -3)$. The set $S = \{\alpha\} \subset W$ is a linearly independent set, i.e., S is also a basis of W . Hence, $\dim(W) = 1$.

Example 3. Let $W = \{(x, y, z) \in \mathbb{R}^3 : x + 3y + 4z = 0\}$. Find the basis and dimension of V .

LECTURE 30 BEGINS:

8 Rank of a Matrix

Let $A = (a_{ij})$, where $1 \leq i, j \leq n$, be a non-zero matrix. Then, the rank of A is said to be $r \in \mathbb{N}$ such that there exists at least one minor (non-zero) of A of order r .

Example 1. Let us consider the matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. Find the rank of A .

Solution. We notice that $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0$. So, $\text{rank}(A) = 3$.

Example 2. Let us consider the matrix $A = \begin{bmatrix} 1 & 0 & 3 \\ 4 & -1 & 5 \\ 2 & 0 & 6 \end{bmatrix}$. Find the rank of A .

Solution. We find that $\det(A) = \begin{vmatrix} 1 & 0 & 3 \\ 4 & -1 & 5 \\ 2 & 0 & 6 \end{vmatrix} = 0$, which implies that $\text{rank}(A) < 3$. Next, we have the minor of order 2, $M = \begin{vmatrix} 1 & 0 \\ 4 & -1 \end{vmatrix} = -1 \neq 0$. Hence, $\text{rank}(A) = 2$.

Example 3. Let us consider the matrix $A = \begin{bmatrix} 2 & 3 & -1 & 1 \\ 3 & 0 & 4 & 2 \\ 6 & 9 & -3 & 3 \end{bmatrix}_{3 \times 4}$. Find the rank of A .

Solution. Since the third row is a multiple of first row, so the rank of A is always less than 3. We take a minor of order 2, $M = \begin{vmatrix} 2 & 3 \\ 3 & 0 \end{vmatrix} = -9 \neq 0 \implies \text{rank}(A) = 2$.

Note. If we have $A = (a_{ij})$, where $1 \leq i \leq m, 1 \leq j \leq n$, then $0 < \text{rank}(A) \leq \min(m, n)$.

9 Row Equivalent or Column Equivalent Matrices

A matrix B is said to be row equivalent to a matrix A , if it is obtained from A by some successive row operations (finite in number). Similarly, we can define column equivalent matrix.

Example 1. Find the rank of $A = \begin{bmatrix} 0 & 0 & 2 & 2 & 0 \\ 1 & 3 & 2 & 4 & 1 \\ 2 & 6 & 2 & 6 & 2 \end{bmatrix}_{3 \times 5}$.

Solution.

$$A = \begin{bmatrix} 0 & 0 & 2 & 2 & 0 \\ 1 & 3 & 2 & 4 & 1 \\ 2 & 6 & 2 & 6 & 2 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 3 & 2 & 4 & 1 \\ 2 & 6 & 2 & 6 & 2 \end{bmatrix} = B_1$$

$$B_1 \xrightarrow[R_3-2R_1]{R_2-2R_1} \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 3 & 0 & 2 & 1 \\ 2 & 6 & 0 & 4 & 2 \end{bmatrix} = B_2 \xrightarrow{R_3-2R_2} \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = B_3 = B.$$

All the matrices B, B_1, B_2, B_3 are row equivalent to A . Note that $\text{rank}(B) = \text{rank}(A)$ as B has a minor of order 2 such that $\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2 \neq 0$.

10 Row-Reduced or Echelon Form

An $m \times n$ matrix A is called row-reduced (Echelon) matrix, if

- (i) the first non-zero element in a non-zero row is 1 (identity element).
- (ii) there is an integer r ($0 \leq r \leq m$) such that the first r rows of A are non-zero rows and the remaining rows (if there is any) are all zero.

- (iii) for each non-zero row, if the leading 1 of the row i occurs in the column k_i , then $k_1 < k_2 < \dots$, and so on.

Example 1. Apply the elementary row operations to reduce the following matrix to a row reduced Echelon matrix.

$$A = \begin{bmatrix} 2 & 0 & 4 & 2 \\ 3 & 2 & 6 & 5 \\ 5 & 2 & 10 & 7 \\ 0 & 3 & 2 & 5 \end{bmatrix}_{4 \times 4}$$

Solution. Row operations to be performed to get the Echelon form

$$B = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{4 \times 4}$$

of the matrix A are (respectively): $\frac{1}{2}R_1, R_2 - 3R_1, R_3 - 5R_1, \frac{1}{2}R_2, R_3 - 2R_2, R_4 - 3R_2, R_{34} (R_3 \leftrightarrow R_4), \frac{1}{2}R_3, R_1 - 2R_3$. Clearly, the rank of B is 3 and so is $\text{rank}(A)$.

LECTURE 31 BEGINS:

Example 2. Find the rank of the matrix given below.

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 4 & 8 & 6 \\ 0 & 0 & 5 & 8 \\ 3 & 6 & 6 & 3 \end{bmatrix}_{4 \times 4}$$

Solution.

$$\begin{aligned} A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 4 & 8 & 6 \\ 0 & 0 & 5 & 8 \\ 3 & 6 & 6 & 3 \end{bmatrix} &\xrightarrow[R_2 - 2R_1]{R_4 - 3R_1} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 6 & 6 \\ 0 & 0 & 5 & 8 \\ 0 & 0 & 3 & 3 \end{bmatrix} \xrightarrow[\frac{1}{6}R_2]{\frac{1}{3}R_4} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 5 & 8 \\ 0 & 0 & 1 & 1 \end{bmatrix} \\ &\xrightarrow[R_4 - R_2]{R_3 - 5R_2} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\frac{1}{3}R_3} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ &\xrightarrow{R_2 - R_3} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 - R_2} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B, \text{ (say).} \end{aligned}$$

B is a row-reduced/row-equivalent matrix to A . B is also a row-reduced Echelon matrix to A . Hence, the rank of A is same as B , i.e., $\text{rank}(A) = 3$.

Theorem 1. If a matrix A is row-equivalent to a row-reduced Echelon matrix/form B having r non-zero rows, then $\text{rank}(A) = r$.

Example 1 (Inverse of a matrix). Find the inverse of $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & 4 \\ 3 & 3 & 7 \end{bmatrix}$.

Solution. We form a 3×6 matrix of type $[A/I_3]$ and perform row operations to reduce A to a row-reduced Echelon matrix.

$$B = [A/I_3] = \begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 \\ 2 & 4 & 4 & 0 & 1 & 0 \\ 3 & 3 & 7 & 0 & 0 & 1 \end{bmatrix}_{3 \times 6} \xrightarrow[R_2-2R_1]{R_3-3R_1} \begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{bmatrix} \xrightarrow{\frac{1}{2}R_2} \begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_1-R_2} \begin{bmatrix} 1 & 0 & 2 & 2 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{bmatrix} \xrightarrow{R_1-2R_3} \begin{bmatrix} 1 & 0 & 0 & 8 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{bmatrix}$$

Hence, the inverse of the matrix A is $C = A^{-1} = \begin{bmatrix} 8 & -\frac{1}{2} & 0 \\ -1 & \frac{1}{2} & 0 \\ -3 & 0 & 1 \end{bmatrix}$.

Example 2. Solve the following system of linear equations.

$$\begin{aligned} x_1 + x_2 &= 4 \\ x_2 + x_3 &= 1 \\ 2x_1 + x_2 + 4x_3 &= 7 \end{aligned}$$

Solution. We can write the above system in the below compact form:

$$A\vec{x} = \vec{b},$$

where $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 2 & 1 & 4 \end{bmatrix}$, $\vec{x} = (x_1, x_2, x_3)$ and $\vec{b} = (4, 1, 7)$.

Let us form the Augmented matrix:

$$\overline{A} = [A/b] = \begin{bmatrix} 1 & 1 & 0 & 4 \\ 0 & 1 & -1 & 1 \\ 2 & 1 & 4 & 7 \end{bmatrix}_{3 \times 4}$$

The steps $R_1 - R_2, R_3 + R_2, \frac{1}{3}R_3, R_1 - R_3, R_2 + R_3$ once applied on A , leads to

$$\overline{A} \equiv \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = C$$

The modified system of equations will be $x_1 = 3, x_2 = 1, x_3 = 0$. Therefore, $(3, 1, 0)$ is a solution of the given system.

Example 3. Verify if the below system has a solution.

$$x + 2y - z = 10$$

$$-x + y + 2z = 2$$

$$2x + y - 3z = 2$$

Solution. The augment matrix,

$$\overline{A} = [A/b] = \begin{bmatrix} 1 & 2 & -1 & 10 \\ -1 & 1 & 2 & 2 \\ 2 & 1 & -3 & 2 \end{bmatrix}$$

The steps $R_2 + R_1, R_3 - 2R_1, \frac{1}{3}R_2, R_1 - 2R_2, R_3 + 3R_2$ once applied on A , leads to

$$\overline{A} \equiv \begin{bmatrix} 1 & 0 & -\frac{5}{3} & 2 \\ 0 & 1 & \frac{1}{3} & 4 \\ 0 & 0 & 0 & -6 \end{bmatrix}$$

The modified system of equations:

$$x - \frac{5}{3}z = 2$$

$$y + \frac{z}{3} = 4$$

$$0 = -6 \text{ (ABSURD)}$$

Hence, the given system has NO solution.

11 Gauss-Elimination Method

Example 1. Solve the following system by Gauss-Elimination method:

$$2x - 2y = -6$$

$$x - y + z = 1$$

$$3y - 2z = -5$$

Solution. The coefficient matrix, $A = \begin{bmatrix} 2 & -2 & 0 \\ 1 & -1 & 1 \\ 0 & 3 & -2 \end{bmatrix}$, $\vec{x} = (x, y, z)$ and $\vec{b} = (-6, 1, -5)$. The augmented matrix:

$$\overline{A} = [A/b] = \begin{bmatrix} 2 & -2 & 0 & -6 \\ 1 & -1 & 1 & 1 \\ 0 & 3 & -2 & -5 \end{bmatrix}$$

Which reduces after applying the operations $\frac{1}{2}R_1, R_2 - R_1, R_2 \leftrightarrow R_3$ to $\tilde{A} = \begin{bmatrix} 1 & -1 & 0 & -3 \\ 0 & 3 & -2 & -5 \\ 0 & 0 & 1 & 4 \end{bmatrix}$.

From the third row, $z = 4$. Back substituting this value in the equation corresponding to 2nd row

leads to, $3y - 2z = -5 \implies y = 1$. Again, we use these values and substitute in the equation corresponding to 1st row and get, $x - y = -3 \implies x = -2$. Hence, the required solution is $(-2, 1, 4)$.

LECTURE 32 BEGINS:

12 Linear Transformation

Let V and W be two vector spaces over a field $\mathbb{F}$. A mapping $T : V \rightarrow W$ is said to be a linear mapping/transformation, if it satisfies the following conditions:

$$(i) \quad T(\alpha + \beta) = T(\alpha) + T(\beta), \quad \forall \alpha, \beta \in V.$$

$$(ii) \quad T(c\alpha) = cT(\alpha), \quad \forall \alpha \in V.$$

In short, $T(c\alpha + \beta) = cT(\alpha) + T(\beta)$, $\forall c \in \mathbb{F}, \alpha, \beta \in V$.

Example 1. The identity mapping $T : V \rightarrow V$ defined by $T(x) = x$, $\forall x \in V$, which clearly satisfies, $T(x + y) = x + y = T(x) + T(y)$ and $T(cx) = cx = cT(x)$.

Example 2. The zero mapping $T : V \rightarrow W$ defined by $T(x) = \theta'$, $\forall x \in V$, $\theta' \in W$. Which follows as the previous example.

Example 3. Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ such that $T(x_1, x_2, x_3) = (x_1, x_2, 0)$, where $(x_1, x_2, x_3) \in \mathbb{R}^3$.

Solution. Let $\alpha = (x_1, x_2, x_3), \beta = (y_1, y_2, y_3) \in \mathbb{R}^3$, then we can easily check that

$$\begin{aligned} T(\alpha + \beta) &= T(x_1 + y_1, x_2 + y_2, x_3 + y_3) \\ &= (x_1 + y_1, x_2 + y_2, 0) \\ &= (x_1, x_2, 0) + (y_1, y_2, 0) \\ &= T(\alpha) + T(\beta), \\ T(c\alpha) &= T(cx_1, cx_2, cx_3) \\ &= (cx_1, cx_2, 0) \\ &= c(x_1, x_2, 0) \\ &= cT(\alpha). \end{aligned}$$

Hence, T is a linear transformation.

Example 4. Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ such that $T(x_1, x_2, x_3) = (x_1 + 1, x_2 + 1, x_3 + 1)$, where $(x_1, x_2, x_3) \in \mathbb{R}^3$. Verify whether T is linear. (Answer: No)

13 Kernel of a Linear Transformation

Let V and W be two vector spaces and $T : V \rightarrow W$ be a linear mapping. The set of all $\alpha \in V$ such that $T(\alpha) = \theta'$ in W , where θ' is the null element in W , is said to be the Kernel of T and it is defined by $\ker(T)$. In other words,

$$\ker(T) = \{\alpha \in V : T(\alpha) = \theta'\}. \quad (2)$$

The dimension of $\ker(T)$ is called the Nullity of T . Also, $\ker(T) \subset V$ is a subspace.

14 Image of a Linear Transformation

Let $T : V \rightarrow W$ be a linear map. The image of the elements of V under the mapping T forms a subset of W . This subset is called the image of T and it is denoted by $\text{Im}(T)$. In other words,

$$\text{Im}(T) = \{T(\alpha) : \alpha \in V\}.$$

Note that $\text{Im}(T) \subset W$ is a subspace. The dimension of $\text{Im}(T)$ is called rank of T , i.e., $\text{rank}(T)$.

Example 1. Verify whether the below transformation is linear or not.

$$T(x_1, x_2, x_3) = (x_1 + x_2 + x_3, 2x_1 + x_2 + 2x_3, x_1 + 2x_2 + x_3), \quad \text{where } (x_1, x_2, x_3) \in \mathbb{R}^3$$

If yes, find $\ker(T)$, $\text{Im}(T)$.

Solution. Let $\alpha = (x_1, x_2, x_3)$ and $\beta = (y_1, y_2, y_3)$. Then,

$$\begin{aligned} T(\alpha + \beta) &= T(x_1 + y_1, x_2 + y_2, x_3 + y_3) \\ &= (x_1 + y_1 + x_2 + y_2 + x_3 + y_3, 2x_1 + 2y_1 + x_2 + y_2 + 2x_3 + 2y_3, x_1 + y_1 + 2x_2 + 2y_2 + x_3 + y_3) \\ &= (x_1 + x_2 + x_3 + y_1 + y_2 + y_3, 2x_1 + x_2 + 2x_3 + 2y_1 + y_2 + 2y_3, x_1 + 2x_2 + x_3 + y_1 + 2y_2 + y_3) \\ &= (x_1 + x_2 + x_3, 2x_1 + x_2 + 2x_3, x_1 + 2x_2 + x_3) + (y_1 + y_2 + y_3, 2y_1 + y_2 + 2y_3, y_1 + 2y_2 + y_3) \\ &= T(x_1, x_2, x_3) + T(y_1, y_2, y_3) = T(\alpha) + T(\beta) \end{aligned}$$

For $c \in \mathbb{F}$ and $\alpha = (x_1, x_2, x_3) \in V$

$$T(c\alpha) = T(cx_1, cx_2, cx_3) = (cx_1 + cx_2 + cx_3, 2cx_1 + cx_2 + 2cx_3, cx_1 + 2cx_2 + cx_3) = cT(\alpha)$$

$\implies T$ is a linear map.

Now let $\alpha = (x_1, x_2, x_3) \in \mathbb{R}^3$. Then, by the definition of Kernel, we write

$$\begin{aligned} T(\alpha) = \theta &\implies T(x_1, x_2, x_3) = (0, 0, 0) \\ \implies (x_1 + x_2 + x_3, 2x_1 + x_2 + 2x_3, x_1 + 2x_2 + x_3) &= (0, 0, 0) \end{aligned}$$

which leads to the system of linear equations

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ 2x_1 + x_2 + 2x_3 &= 0 \\ x_1 + 2x_2 + x_3 &= 0. \end{aligned}$$

which further reduces to: $x_1 + x_3 = 0, x_2 = 0$.

The solution to the above equations is $k(1, 0, -1)$, $k \in \mathbb{R}$. Hence, $\ker(T) = \{(1, 0, -1)\}$ and $\dim(\ker(T)) = 1$, also known as Nullity of T .

LECTURE 33 BEGINS:

Let us choose one of basis of $\mathbb{R}$ having the elements: $\alpha_1 = (1, 0, 0), \alpha_2 = (0, 1, 0), \alpha_3 = (0, 0, 1)$. From which one can easily get: $T(1, 0, 0) = (1, 2, 1) = T(\alpha_1)$, $T(0, 1, 0) = (1, 1, 2) = T(\alpha_2)$, $T(0, 0, 1) = (1, 2, 1) = T(\alpha_3)$. Clearly, the set $\{T(\alpha_1), T(\alpha_2), T(\alpha_3)\}$ is linearly dependent. Then, we consider

the set $\{T(\alpha_1), T(\alpha_2)\} = \{(1, 2, 1), (1, 1, 2)\}$, which is linearly independent. The image set $\text{Im}(T)$ is generated/spanned by $\{T(\alpha_1), T(\alpha_2)\}$, i.e., $\text{Im}(T) = \{T(\alpha_1), T(\alpha_2)\}$. Also, $\dim(\text{Im}(T)) = 2$. From here, we observe that: $\dim(\ker(T)) + \dim(\text{Im}(T)) = 1 + 2 = 3 = \dim(\mathbb{R}^3) \implies \text{null}(T) + \text{rank}(T) = \dim(V)$.

Theorem 1. If $T : V \rightarrow W$ is a linear transformation, then $\text{null}(T) + \text{rank}(T) = \dim(V)$. This theorem is known as Rank-Nullity Theorem.

Theorem 2. Let V and W be two vector spaces over a field $\mathbb{F}$, and $T : V \rightarrow W$ be a linear transformation. Let $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be a basis of V , then $\{T(\alpha_1), T(\alpha_2), \dots, T(\alpha_n)\}$ generates $\text{Im}(T)$.

Example 2. Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^4$ be such that $T(x_1, x_2, x_3) = (x_2 + x_3, x_3 + x_1, x_1 + x_2, x_1 + x_2 + x_3)$.

- (i) Verify T is linear.
- (ii) Find $\ker(T)$.
- (iii) Verify $\{T(1, 0, 0), T(0, 1, 0), T(0, 0, 1)\}$ is linearly independent set.

Solution. (i) Let us consider $\alpha = (x_1, x_2, x_3)$ and $\beta = (y_1, y_2, y_3)$. So,

$$\begin{aligned} T(\alpha + \beta) &= T(x_1 + y_1, x_2 + y_2, x_3 + y_3) \\ &= (x_2 + y_2 + x_3 + y_3, x_3 + y_3 + x_1 + y_1, x_1 + y_1 + x_2 + y_2, x_1 + y_1 + x_2 + y_2 + x_3 + y_3) \\ &= (x_2 + x_3 + y_2 + y_3, x_3 + x_1 + y_3 + y_1, x_1 + x_2 + y_1 + y_2, x_1 + x_2 + x_3 + y_1 + y_2 + y_3) \\ &= T(x_1, x_2, x_3) + T(y_1, y_2, y_3) = T(\alpha) + T(\beta) \end{aligned}$$

Similarly, one can check that $T(c\alpha) = cT(\alpha) \implies T$ is linear.

(ii) By definition, $\ker(T) = \{\alpha = (x_1, x_2, x_3) \in \mathbb{R}^3 : T(\alpha) = \vec{0} \in \mathbb{R}^4\}$. This leads to a system of equations: $x_2 + x_3 = 0$, $x_3 + x_1 = 0$, $x_1 + x_2 = 0$, $x_1 + x_2 + x_3 = 0$, which has the solution: $x_1 = 0, x_2 = 0, x_3 = 0$. Therefore, $\ker(T) = \{(0, 0, 0)\} = \{\theta\}$.

(iii) Now let $\alpha_1 = (1, 0, 0), \alpha_2 = (0, 1, 0), \alpha_3 = (0, 0, 1)$. Then, $T(1, 0, 0) = (0, 1, 1, 1), T(0, 1, 0) = (1, 0, 1, 1), T(0, 0, 1) = (1, 1, 0, 1)$. Let $c_1, c_2, c_3 \in \mathbb{F}$ be such that

$$\begin{aligned} c_1 T(\alpha_1) + c_2 T(\alpha_2) + c_3 T(\alpha_3) &= \vec{0} \\ \implies c_1(0, 1, 1, 1) + c_2(1, 0, 1, 1) + c_3(1, 1, 0, 1) &= (0, 0, 0, 0) \end{aligned}$$

leading to: $c_2 + c_3 = 0$, $c_1 + c_3 = 0$, $c_1 + c_2 = 0$, $c_1 + c_2 + c_3 = 0$. So, $c_1 = 0, c_2 = 0, c_3 = 0$. Clearly, all c_i 's, $i = 1, 2, 3$ are zero. The set $\{T(\alpha_1), T(\alpha_2), T(\alpha_3)\}$ is linearly independent.

Example 3. Let us consider a map $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be such that $T(x_1, x_2, x_3) = (x_1 + x_2, x_3 + x_2)$.

- (i) Verify if T is linear.
- (ii) If yes, find $\ker(T)$ and $\text{Im}(T)$.

Example 4. Let $T : \mathbb{R}^7 \rightarrow \mathbb{R}^8$ be such that $T(x_1, x_2, \dots, x_7) = (x_1 + x_2, x_2 + x_3, \dots, x_7 + x_8, x_8 + x_1)$.

- (i) Verify if T is linear.
- (ii) If yes, find $\ker(T)$ and $\text{Im}(T)$.

LECTURE 34 BEGINS:

15 Matrix Representation of a Linear Map

Let V and W be two vector spaces, and $T : V \rightarrow W$ is a linear map. Let $(\alpha_1, \alpha_2, \dots, \alpha_n)$ be the ordered basis in V and $(\beta_1, \beta_2, \dots, \beta_m)$ be the ordered basis in W . Then, each $T(\alpha_i)$ can be written as the linear combination of β_i 's, $\forall i$, i.e.,

$$\begin{aligned} y_1 &= T(\alpha_1) = a_{11}\beta_1 + a_{21}\beta_2 + \dots + a_{m1}\beta_m \\ y_2 &= T(\alpha_2) = a_{12}\beta_1 + a_{22}\beta_2 + \dots + a_{m2}\beta_m \\ &\vdots \\ y_n &= T(\alpha_n) = a_{1n}\beta_1 + a_{2n}\beta_2 + \dots + a_{mn}\beta_m \end{aligned}$$

where a_{ij} , $\forall i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$ can be uniquely determined. then, the corresponding matrix A is given by

$$A_{m \times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

We see that $y = A^t B$, where $B = (\beta_1, \beta_2, \dots, \beta_m)$.

Example 1. A linear map $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ is given by $T(x_1, x_2, x_3) = (3x_1 - 2x_2 + x_3, x_1 - 3x_2 - 2x_3)$. Find the matrix representation of T with respect to the ordered/standard basis in $\mathbb{R}^3$.

Solution. Lets us take the ordered basis in $\mathbb{R}^3$ as $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$. We now write:

$$\begin{aligned} T(\alpha_1) &= T(1, 0, 0) = (3, 1) \\ T(\alpha_2) &= T(0, 1, 0) = (-2, -3) \\ T(\alpha_3) &= T(0, 0, 1) = (1, -2) \end{aligned}$$

The corresponding matrix representation of T will be: $A = \begin{bmatrix} 3 & -2 & 1 \\ 1 & -3 & -2 \end{bmatrix}_{2 \times 3}$.

Example 2. For the same linear transformation as the previous example, find the matrix representation of T with respect to the ordered basis $\{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$ of $\mathbb{R}^3$ and $\{(1, 0), (0, 1)\}$ of $\mathbb{R}^2$.

Solution. We have: $T(1, 1, 0) = (1, -2)$, $T(1, 0, 1) = (4, -1)$, $T(0, 1, 1) = (-1, -5)$. So, we can write:

$$\begin{aligned} T(1, 1, 0) &= (1, -2) = 1(1, 0) - 2(0, 1) \\ T(1, 0, 1) &= (4, -1) = 4(1, 0) - 1(0, 1) \\ T(0, 1, 1) &= (-1, -5) = -1(1, 0) - 5(0, 1) \end{aligned}$$

and hence, the matrix representation will be $A = \begin{bmatrix} 1 & 4 & -1 \\ -2 & -1 & -5 \end{bmatrix}_{2 \times 3}$.

Example 3. Let the transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be given by $T(x_1, x_2, x_3) = (2x_1 + x_2 - x_3, x_2 +$

$4x_3, x_1 - x_2 + 3x_3$). Find the matrix representation of T with respect to the ordered basis in $\mathbb{R}^3$.

Solution. Choose the basis: $S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ and proceed as previous examples.

Example 4. The matrix representation of a linear map $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ with respect to the ordered basis $\{(0, 1, 1), (1, 0, 1), (1, 1, 0)\}$ of $\mathbb{R}^3$ is given by $A = \begin{bmatrix} 0 & 3 & 0 \\ 2 & 3 & -2 \\ 2 & -1 & 2 \end{bmatrix}$. Find the linear transformation of A .

Solution. We observe that:

$$T(0, 1, 1) = 0(0, 1, 1) + 2(1, 0, 1) + 2(1, 1, 0) = (4, 2, 2)$$

$$T(1, 0, 1) = 3(0, 1, 1) + 3(1, 0, 1) - (1, 1, 0) = (2, 2, 6)$$

$$T(1, 1, 0) = 0(0, 1, 1) - 2(1, 0, 1) + 2(1, 1, 0) = (0, 2, -2)$$

Let (x, y, z) be in $\mathbb{R}^3$ such that $(x, y, z) = c_1(0, 1, 1) + c_2(1, 0, 1) + c_3(1, 1, 0)$. Which gives: $c_1 = \frac{y+z-x}{2}$, $c_2 = \frac{x+z-y}{2}$, $c_3 = \frac{x+y-z}{2}$.

Therefore,

$$\begin{aligned} (x, y, z) &= \frac{y+z-x}{2}(0, 1, 1) + \frac{x+z-y}{2}(1, 0, 1) + \frac{x+y-z}{2}(1, 1, 0) \\ T(x, y, z) &= \frac{y+z-x}{2}T(0, 1, 1) + \frac{x+z-y}{2}T(1, 0, 1) + \frac{x+y-z}{2}T(1, 1, 0) \\ &= \frac{y+z-x}{2}(4, 2, 2) + \frac{x+z-y}{2}(2, 2, 6) + \frac{x+y-z}{2}(0, 2, -2) \\ &= (-x+y+3z, x+y+z, x-3y+5z) \end{aligned}$$

which certainly is the required linear map.

LECTURE 35 BEGINS:

16 Eigenvalues and Eigenvectors

Let A be an $n \times n$ matrix and $\lambda \in \mathbb{F}$. Then, $\det(A - \lambda I)$ is said to be a Characteristic Polynomial of A where $I_{n \times n}$ is the identity matrix, and is denoted by $\Psi_\lambda(A)$. Furthermore, $\Psi_\lambda(A) = 0$ is called the Characteristic Equation. The roots of $\Psi_\lambda(A) = 0$ are called the Eigenvalues of the matrix A . If a root of $\Psi_\lambda(A) = 0$ has multiplicity r , then it is said to be an r -fold eigenvalue. For an r -fold eigenvalue λ , r is called the Algebraic Multiplicity of λ , and the rank of characteristic subspace corresponding to λ is called the Geometric Multiplicity.

Example 1. Find the eigenvalue of the matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Solution. Let $\lambda \in \mathbb{F}$. Then, the characteristic polynomial is, $\det(A - \lambda I) = \begin{vmatrix} 0-\lambda & 1 \\ 1 & 0-\lambda \end{vmatrix} = \lambda^2 - 1$. Therefore, the characteristic equation is given by $\det(A - \lambda I) = 0 \implies \lambda^2 - 1 = 0 \implies \lambda = -1, 1$. The required eigenvalues are -1 and 1 .

Example 2. Find the eigenvalues of $A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$.

Solution. Let $\lambda \in \mathbb{F}$. Then, the characteristic polynomial is given by

$$\Psi_\lambda(A) = \det(A - \lambda I) = \det \left\{ \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\} = \begin{vmatrix} 1-\lambda & 1 & 1 \\ -1 & -1-\lambda & -1 \\ 0 & 0 & 1-\lambda \end{vmatrix}.$$

The characteristic equation is $\Psi_\lambda(A) = 0 \implies \begin{vmatrix} 1-\lambda & 1 & 1 \\ -1 & -1-\lambda & -1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$. After solving the

determinant, we get $(1-\lambda)\lambda^2 = 0 \implies \lambda = 0, 0, 1$. Clearly, 0 is an eigenvalue with the algebraic multiplicity 2 and 1 has the algebraic multiplicity 1.

Now let $\lambda = \lambda_1$ be an eigenvalue of $A_{n \times n}$. Then, there exists a non-zero vector $X \in \mathbb{R}^n$ such that $AX = \lambda_1 X$.

Example 3. Find the eigenvectors corresponding to the eigenvalues we found in the previous example.

Solution. We are given $A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$. We know that $\lambda = 0, 0, 1$. Let $X = (x, y, z) \in \mathbb{R}^3$

be the eigenvector corresponding to $\lambda = 0$. Then,

$$\begin{aligned} AX &= \lambda X \\ \implies \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= 0 \begin{bmatrix} x \\ y \\ z \end{bmatrix} \\ \implies \begin{bmatrix} x+y+z \\ -x-y-z \\ z \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

We get: $z = 0, x + y = 0 \implies x = -y = -k$, where $k \in \mathbb{R}$. So, the required eigenvector corresponding to the eigenvalue 0 is: $X = k(-1, 1, 0)$, $k \in \mathbb{R}$.

Now let $\lambda = 1$. Then, the eigenvector can be obtain

$$\begin{aligned} AX &= \lambda X \\ \implies \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= 1 \begin{bmatrix} x \\ y \\ z \end{bmatrix} \\ \implies \begin{bmatrix} x+y+z \\ -x-y-z \\ z \end{bmatrix} &= \begin{bmatrix} x \\ y \\ z \end{bmatrix} \end{aligned}$$

We get: $y + z = 0, x + y = -(y + z) = 0$. So, we get $x = z = k \in \mathbb{R}$. The required eigenvector corresponding to the eigenvalue 1 is: $X = k(1, -1, 1)$, $k \in \mathbb{R}$.

LECTURE 36 BEGINS:

Example 4. Consider $A = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}$. Find the characteristic equation.

Solution. Let $\lambda \in \mathbb{F}$. Then, the characteristic equation is $\det(A - \lambda I_2) = 0 \implies \begin{vmatrix} 2-\lambda & 1 \\ 3 & 5-\lambda \end{vmatrix} = 0 \implies \lambda^2 - 7\lambda + 7 = 0$.

Cayley-Hamilton Theorem. Every square matrix A of order n satisfies its own characteristic equation.

Example 5. Verify the Cayley-Hamilton theorem for the matrix $A = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}$.

Solution. The characteristic equation for A is given by $\lambda^2 - 7\lambda + 7 = 0$ as we found out previously.
Goal: $A^2 - 7A + 7I = \vec{0}$

$$\begin{aligned} \text{L.H.S.} &\implies A \cdot A - 7A + 7I \\ &\implies \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix} - 7 \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix} + 7 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &\implies \begin{bmatrix} 7 & 7 \\ 14 & 28 \end{bmatrix} - \begin{bmatrix} 14 & 7 \\ 21 & 35 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} \\ &\implies \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \implies \mathbf{0}_{2 \times 2} \implies \text{R.H.S.} \end{aligned}$$

Example 6. Using the Cayley-Hamilton theorem, find the inverse of $\begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}$.

Solution. We know $\begin{vmatrix} 2 & 1 \\ 3 & 5 \end{vmatrix} = 7 > 0$. Again,

$$\begin{aligned} A^2 - 7A + 7I &= 0 \\ \implies A(A - 7I) &= -7I \\ \implies A^{-1} &= -\frac{1}{7}(A - 7I) \\ \implies A^{-1} &= \frac{1}{7} \left(\begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix} \right) \\ \implies A^{-1} &= \frac{1}{7} \begin{bmatrix} 5 & -1 \\ -3 & 2 \end{bmatrix}. \end{aligned}$$

Proposition 1. If A is a singular matrix, then 0 is an eigenvalue of A .

Proposition 2. The eigenvalues in a diagonal matrix are its diagonal entries.

Proposition 3. If λ is an eigenvalue of a non-singular matrix, then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1} .

Eigenvector. Let A be an $n \times n$ matrix over a field $\mathbb{F}$. A non-null vector X belonging to $V_n(k)$ is said to be an eigenvector of A , if there exists a scalar $\lambda \in \mathbb{F}$ such that $AX = \lambda X$ holds.

Example 7. Let $A = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix}$. Find the eigenvalues and eigenvectors of A .

Solution. Let $\lambda \in \mathbb{F}$. Then, $\det(A - \lambda I) = \begin{vmatrix} 1 - \lambda & 3 \\ 4 & 5 - \lambda \end{vmatrix} = 0 \implies \lambda = -1, 7$.

Let $X \in \mathbb{R}^2$ be the eigenvector corresponding to the eigenvalue -1, then

$$\begin{aligned} AX &= \lambda X \\ \implies \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= -1 \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ \implies \begin{bmatrix} x_1 + 3x_2 \\ 4x_1 + 5x_2 \end{bmatrix} &= - \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ \implies \begin{bmatrix} 2x_1 + 3x_2 \\ 4x_1 + 6x_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

We get $x_1 = -\frac{3}{2}c$, where $x_2 = c \in \mathbb{R}$. Then, the required eigenvector is $X = \begin{bmatrix} -\frac{3}{2} \\ 1 \end{bmatrix} c ; c \in \mathbb{F}$.

For $\lambda = 7$, we have

$$\begin{aligned} AX &= \lambda X \\ \implies \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= 7 \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ \implies \begin{bmatrix} x_1 + 3x_2 \\ 4x_1 + 5x_2 \end{bmatrix} &= \begin{bmatrix} 7x_1 \\ 7x_2 \end{bmatrix} \\ \implies \begin{bmatrix} -6x_1 + 3x_2 \\ 4x_1 - 2x_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

We get $x_1 = \frac{1}{2}c$, where $x_2 = c \in \mathbb{R}$. Then, the required eigenvector is $X = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} c ; c \in \mathbb{F}$.

LECTURE 37 BEGINS:

Example 8. Let $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Find the eigenvalues and eigenvectors of A .

Solution. Let $\lambda \in \mathbb{F}$. Then, the characteristic equation is: $\det(A - \lambda I) = 0 \implies \begin{vmatrix} 0 - \lambda & -1 \\ 1 & 0 - \lambda \end{vmatrix} \implies$

$$\lambda^2 + 1 = 0 \implies \lambda = \pm \iota.$$

Case 1. Let $\lambda = \iota$ and $X = (x_1, x_2) \in \mathbb{R}^2$. Then

$$\begin{aligned} AX &= \lambda X \\ \implies \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \iota \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ \implies \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} &= \begin{bmatrix} \iota x_1 \\ \iota x_2 \end{bmatrix} \\ \implies \begin{bmatrix} -\iota x_1 - x_2 \\ x_1 - \iota x_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

For eigenvalue $\lambda = \iota$, we get $x_1 = \iota c$, where $x_2 = c \in \mathbb{R}$. Then, the required eigenvector is $X = \begin{bmatrix} \iota \\ 1 \end{bmatrix} c$; $c \in \mathbb{F}$.

Case 2. Let $\lambda = -\iota$ and $X = (x_1, x_2) \in \mathbb{R}^2$. Then

$$\begin{aligned} AX &= \lambda X \\ \Rightarrow \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= -\iota \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ \Rightarrow \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} &= -\begin{bmatrix} \iota x_1 \\ \iota x_2 \end{bmatrix} \\ \Rightarrow \begin{bmatrix} \iota x_1 - x_2 \\ x_1 + \iota x_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

For eigenvalue $\lambda = -\iota$, we get $x_1 = -\iota c$, where $x_2 = c \in \mathbb{R}$. Then, the required eigenvector is $X = \begin{bmatrix} -\iota \\ 1 \end{bmatrix} c$; $c \in \mathbb{F}$.

17 Diagonalisation of a Matrix

Similar Matrices. Let us consider all $n \times n$ matrices over the field $\mathbb{F}$. An $n \times n$ matrix A is said to be similar to an $n \times n$ matrix B , if there exists a non-singular matrix P of order $n \times n$ such that $B = P^{-1}AP$. We say that A and B are similar matrices.

$$\begin{aligned} B &= P^{-1}AP \\ \Rightarrow PB &= PP^{-1}AP = (PP^{-1})AP = IAP = AP \\ \Rightarrow PBP^{-1} &= APP^{-1} = A \\ \Rightarrow A &= Q^{-1}BQ, \text{ where } Q = P^{-1}. \end{aligned}$$

Let $A = (a_{ij})_{n \times n}$ be a square matrix. Then, A is said to be Diagonalisable, if A is similar to an $n \times n$ diagonal matrix.

Example 1. Diagonalise the matrix $A = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$

Solution. Step 1. Let $\lambda \in \mathbb{F}$ be the eigenvalue. Then

$$\begin{aligned} \det(A - \lambda I) &= 0 \\ \Rightarrow \begin{vmatrix} 1 - \lambda & -3 & 3 \\ 3 & -5 - \lambda & 3 \\ 6 & -6 & 4 - \lambda \end{vmatrix} &= 0 \\ \Rightarrow \lambda &= -2, 4, -2. \end{aligned}$$

Step 2. Let $X = (x_1, x_2, x_3) \in \mathbb{R}^3$ be the eigenvector.

$$\begin{aligned}
 AX &= \lambda X \\
 \Rightarrow \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= -2 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\
 \Rightarrow \begin{bmatrix} x_1 - 3x_2 + 3x_3 \\ 3x_1 - 5x_2 + 3x_3 \\ 6x_1 - 6x_2 + 4x_3 \end{bmatrix} &= -2 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\
 \Rightarrow \begin{bmatrix} 3x_1 - 3x_2 + 3x_3 \\ 3x_1 - 3x_2 + 3x_3 \\ 6x_1 - 6x_2 + 6x_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.
 \end{aligned}$$

Which gives us $x_1 - x_2 + x_3 = 0$. So, $x_2 = c$, $x_3 = d$, $x_1 = c - d$, where $c, d \in \mathbb{F}$. The required solution: $X = (c - d, c, d)$. The eigenvector corresponding to $\lambda = -2$ is $(1, 1, 0)$ and $(-1, 0, 1)$, i.e.,

$$\left\{ \begin{bmatrix} c - d \\ c \\ d \end{bmatrix} = c \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} ; c, d \in \mathbb{F} \right\}.$$

Step 3. For $\lambda = 4$, follow the same process and find the eigenvector. it will be continued in the next lecture.

LECTURE 38 BEGINS:

Example 2. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ are similar matrices.

Proposition 1. Let $A = (a_{ij})_{n \times n}$ with eigenvalues $d_1, d_2, \dots, d_n \in \mathbb{F}$, which are not necessarily distinct. Let $D = \text{diag}(d_1, d_2, \dots, d_n)$. Then, for any matrix P , $AP = PD$ holds if and only if the i th column vector of P be an eigenvalue of A corresponding to d_i .

Example 1. (continued...) Diagonalise the matrix $A = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$.

Solution. *Step 3.* For $\lambda = 4$, let $X = (x_1, x_2, x_3) \in \mathbb{R}^3$ be the eigenvector.

$$\begin{aligned}
 AX &= \lambda X \\
 \Rightarrow \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= 4 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\
 \Rightarrow \begin{bmatrix} x_1 - 3x_2 + 3x_3 \\ 3x_1 - 5x_2 + 3x_3 \\ 6x_1 - 6x_2 + 4x_3 \end{bmatrix} &= 4 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\
 \Rightarrow \begin{bmatrix} -3x_1 - 3x_2 + 3x_3 \\ 3x_1 - 9x_2 + 3x_3 \\ 6x_1 - 6x_2 + 0x_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.
 \end{aligned}$$

From here we get $x_1 = x_2$, $x_1 + x_2 = x_3$ and $x_1 + x_3 = 3x_2$. If $x_1 = c = x_2$, where $c \in \mathbb{F}$, then $x_3 = 2c$. Therefore, the required eigenvector corresponding to the eigenvalue 4 is $c \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, $c \in \mathbb{F}$.

Step 4. The required matrix $P = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$. Now we need to check whether $\det(P)$ is 0 or not, i.e., whether the matrix P is non-singular or not. It is easy to find out that the determinant comes out to be a non-zero value.

Next, we check $D = P^{-1}AP$ holds or not, where $P = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$. We use the Proposition 1 and check $DP = AP$

Example 3. Diagonalise $A = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$.

Solution. *Step 1.* Let $\lambda \in \mathbb{F}$ be the eigenvalue. Then

$$\begin{aligned} \det(A - \lambda I) &= 0 \\ \Rightarrow \begin{vmatrix} 1-\lambda & 1 & -2 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{vmatrix} &= 0 \\ \Rightarrow \lambda &= -1, 1, 2. \end{aligned}$$

Step 2. The eigenvectors corresponding to $\lambda = 1, 2, -1$ are $c_1 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$, $c_1 \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$, $c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, respectively, where $c_1, c_2, c_3 \in \mathbb{F}$. Hence, the required matrix $P = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 3 & 0 \\ 1 & 1 & 1 \end{bmatrix}$. Finally, we consider

$$\begin{aligned} AP &= \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 1 \\ 2 & 3 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 6 & 0 \\ 1 & 2 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 1 & 1 \\ 2 & 3 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \\ &= PD, \quad \text{where } D = \text{diag}(1, 2, -1). \end{aligned}$$

LECTURE 39 BEGINS:

18 Orthogonal Matrix

Let $A = (a_{ij})_{n \times n}$ be a square matrix. Then, A is said to be an Orthogonal Matrix, if $AA^T = I_n$.

Proposition 1. If A is orthogonal, then A is non-singular.

We can see $AA^T = I_n \implies |A||A^T| = 1 \implies |A|^2 = 1$. Hence, $\det(A) \neq 0$.

Proposition 2. If A is orthogonal, then $A^{-1} = A^T$.

Proposition 3. If A is orthogonal, then A^{-1} is also orthogonal.

19 Complex Matrix

A matrix A , whose elements are taken from $x \in \mathbb{C}$, is called a complex matrix. This means that A can be expressed as $P + \iota Q$, where P and Q are real matrices. Then, $A = P + \iota Q$, $\bar{A} = P - \iota Q$. Some properties of the complex matrices are given below.

(i) $\bar{\bar{A}} = A$.

(ii) A and B are two complex matrices, then $\overline{AB} = \bar{A}\bar{B}$.

(iii) For a complex matrix, $(\bar{A})^T = \overline{A^T}$. The transpose of $\bar{A}$ is denoted by A° , $(\bar{A})^T = A^\circ$.

20 Hermitian and Skew-Hermitian Matrices

A complex matrix is said to be Hermitian Matrix if $A = A^\circ$, $A^\circ = (\bar{A})^T$. A complex matrix is said to be a Skew-Hermitian Matrix if $A = -A^\circ$, $A^\circ = -(\bar{A})^T$.

Example 1. Check the matrix $A = \begin{bmatrix} 1 & 2 & 3\iota \\ 2 & 0 & 4 \\ -3\iota & 4 & 2 \end{bmatrix}$ is Hermitian or Skew-Hermitian.

Solution. We get $\bar{A} = \begin{bmatrix} 1 & 2 & -3\iota \\ 2 & 0 & 4 \\ 3\iota & 4 & 2 \end{bmatrix}$ and then $(\bar{A})^T = \begin{bmatrix} 1 & 2 & 3\iota \\ 2 & 0 & 4 \\ -3\iota & 4 & 2 \end{bmatrix} = A$. Thus, A is a Hermitian matrix.

Example 2. Check the matrix $A = \begin{bmatrix} 0 & 2 & \iota \\ -2 & 0 & 1 + \iota \\ \iota & -1 + \iota & 0 \end{bmatrix}$ is Hermitian or Skew-Hermitian.

Solution. We get $\bar{A} = \begin{bmatrix} 0 & 2 & -\iota \\ -2 & 0 & 1 - \iota \\ -\iota & -1 - \iota & 0 \end{bmatrix}$ and then $(\bar{A})^T = \begin{bmatrix} 0 & -2 & -\iota \\ 2 & 0 & -1 - \iota \\ -\iota & 1 - \iota & 0 \end{bmatrix} = -A$.

Thus, A is a Skew-Hermitian matrix.

21 Unitary Matrix

A matrix A is said to be a Unitary Matrix if $A\bar{A}^T = I$.

Example 1. Let $A = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}$, where $\omega = \frac{-1 \pm \sqrt{3}\iota}{2}$ and the relations $1 + \omega + \omega^2 = 0$,

$\omega^3 = 1$ hold. Check whether A is Unitary matrix or not.

Theorem 1. If $H = P + \iota Q$ be a Hermitian matrix, then

- (i) the diagonal elements of H are all real numbers.
- (ii) P is a real Symmetric matrix and Q is a real Skew-Symmetric matrix.

Theorem 2. If $S = M + \iota N$ be a Skew-Hermitian matrix, then

- (i) the diagonal elements of S are either 0 or imaginary numbers.
- (ii) M is a real Skew-Symmetric matrix and N is a real Symmetric matrix.

LECTURE 40 BEGINS:

22 Equivalent Matrices and Elementary Matrices

An $n \times n$ matrix B is said to be equivalent to an $n \times n$ matrix A over the same field $\mathbb{F}$ if B can be obtained from A by a finite number of elementary rows and columns operations.

Definition. For a given $m \times n$ matrix A , a row-reduced echelon matrix B can be obtained by applying on A a finite number of elementary row operation on A . On B , we can apply column operations to reduce it into a column reduced echelon matrix, say C . This C has the following properties:

- (a) No zero row is followed by a non-zero row.
- (b) No zero column is followed by a non-zero column.
- (c) The leading 1 in each non-zero row is the only non-zero element.
- (d) The leading 1 in each non-zero column is the only non-zero element.
- (e) The leading 1 in the k th row is the leading 1 in the k th column.

Thus, C has the form $C = \begin{bmatrix} I_r & O_{r,n-r} \\ O_{m-r,r} & O_{m-r,n-r} \end{bmatrix}$, where I is the identity matrix of order r , O_{pq} is the zero matrix of order $p \times q$. Then, C is called the Fully-Reduced-Normal-Form.

Example 1. Obtain the fully reduced normal form of the matrix A give by $A = \begin{bmatrix} 0 & 0 & 1 & 2 & 1 \\ 1 & 3 & 1 & 0 & 3 \\ 2 & 6 & 4 & 2 & 8 \\ 3 & 9 & 4 & 2 & 10 \end{bmatrix}$.

Solution. Let us first apply the elementary row operations.

$$\begin{aligned}
A &= \begin{bmatrix} 0 & 0 & 1 & 2 & 1 \\ 1 & 3 & 1 & 0 & 3 \\ 2 & 6 & 4 & 2 & 8 \\ 3 & 9 & 4 & 2 & 10 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 3 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 & 1 \\ 2 & 6 & 4 & 2 & 8 \\ 3 & 9 & 4 & 2 & 10 \end{bmatrix} \xrightarrow[R_3 - 2R_1]{R_4 - 3R_1} \begin{bmatrix} 1 & 3 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 1 & 2 & 1 \end{bmatrix} \\
&\xrightarrow[R_1 - R_2, R_4 - R_2]{R_3 - 2R_2} \begin{bmatrix} 1 & 3 & 0 & -2 & 2 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{-\frac{1}{2}R_3} \begin{bmatrix} 1 & 3 & 0 & -2 & 2 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_1 + 2R_3]{R_2 - 2R_3} \begin{bmatrix} 1 & 3 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\
&= B \rightarrow \text{row equivalent to } A \\
&\xrightarrow[c_2 - 3c_1]{c_5 - 2c_1} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[c_{34}]{c_{23}} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\
&= C = \begin{bmatrix} I_3 & O_{3,2} \\ O_{1,3} & O_{1,2} \end{bmatrix} \rightarrow \text{fully reduced normal form.}
\end{aligned}$$

Example 2. Show that the matrix $A = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 3 & 0 \\ 6 & 2 & -3 \end{bmatrix}$ is non-singular and express it as the product of elementary matrices.

Solution. We start the process and apply operation as follows:

$$\begin{aligned}
A &= \begin{bmatrix} 2 & 0 & 1 \\ 3 & 3 & 0 \\ 6 & 2 & -3 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 3 & 3 & 0 \\ 6 & 2 & -3 \end{bmatrix} \xrightarrow[R_3 - 6R_1]{R_2 - 3R_1} \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 3 & -\frac{3}{2} \\ 0 & 2 & 0 \end{bmatrix} \\
&\xrightarrow{\frac{1}{3}R_2} \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 2 & 0 \end{bmatrix} \xrightarrow{R_3 - 2R_2} \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_1 - \frac{1}{2}R_3]{R_2 + \frac{1}{2}R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3.
\end{aligned}$$

Clearly, A is row-equivalent to I_3 . Since, $|I_3| \neq 0$, this implies A is non-singular. In short,

$$\begin{aligned}
I_3 &= \left[\left(R_2 + \frac{1}{2}R_3 \right) \left(R_1 - \frac{1}{2}R_3 \right) (R_3 - 2R_2) \left(\frac{1}{3}R_2 \right) (R_2 - 3R_1) (R_3 - 6R_1) \left(\frac{1}{2}R_1 \right) \right] A \\
&= \left[E_{23} \left(\frac{1}{2} \right) E_{13} \left(-\frac{1}{2} \right) E_{32} (-2) E_2 \left(\frac{1}{3} \right) E_{21} (-3) E_{31} (-6) E_1 \left(\frac{1}{2} \right) \right] A \\
&= \left[\left(R_2 + \frac{1}{2}R_3 \right) \left(R_1 - \frac{1}{2}R_3 \right) (R_3 - 2R_2) \left(\frac{1}{3}R_2 \right) (R_2 - 3R_1) (R_3 - 6R_1) \left(\frac{1}{2}R_1 \right) \right] A
\end{aligned}$$

$$\begin{aligned}
A &= \left[E_{23} \left(\frac{1}{2} \right) E_{13} \left(-\frac{1}{2} \right) E_{32} (-2) E_2 \left(\frac{1}{3} \right) E_{21} (-3) E_{31} (-6) E_1 \left(\frac{1}{2} \right) \right]^{-1} I_3 \\
&= \left\{ E_1 \left(\frac{1}{2} \right) \right\}^{-1} \{ E_{31} (-6) \}^{-1} \{ E_{21} (-3) \}^{-1} \left\{ E_2 \left(\frac{1}{3} \right) \right\}^{-1} \{ E_{32} (-2) \}^{-1} \left\{ E_{13} \left(-\frac{1}{2} \right) \right\}^{-1} \left\{ E_{23} \left(\frac{1}{2} \right) \right\}^{-1} I \\
&= \left[E_1 (2) E_{31} (6) E_{21} (3) E_2 (3) E_{32} (2) E_{13} \left(\frac{1}{2} \right) E_{23} \left(-\frac{1}{2} \right) \right] I.
\end{aligned}$$

Definition. An $n \times n$ matrix obtained by applying a single elementary row operations on I_n is said to be an Elementary Matrix of order n . There are three types of elementary matrices.

1. The elementary matrix obtained by applying R_{ij} on I_n is denoted by E_{ij} .
2. The elementary matrix obtained by applying cR_{ij} on I_n is denoted by $E_i(c)$.
3. The elementary matrix obtained by applying $R_i + cR_j$ on I_n is denoted by $E_{ij}(c)$.

$$E_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, E_2(c) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_{13}(c) = \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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